Reconstructing Approximate Tree Metrics

Ittai Abraham, Hebrew University
Mahesh Balakrishnan, Cornell University
Fabian Kuhn, ETH Zurich
Dahlia Malkhi, Microsoft Research
Venugopalan Ramasubramanian, Microsoft Research
Kunal Talwar, Microsoft Research
Introduction

- **Sequoia**: Project at MSR Silicon Valley

- **Goals**: Distributed system to predict network latencies and provide latency-related services by mapping the network nodes to a tree.

- Latency-enabled functionality:
  - Latency prediction
  - Closest node discovery
  - Locality-based clustering
  - Detour routing
  - etc…
**Goals**

Internet Topology is not directly known to end-hosts

Inter-node Ping latencies are available

End-Host Latencies

Model

nXn Estimate Matrix

Clustering, Closest Node Discovery, ...

Embedding
Current State of the Art

- Coordinate-based latency prediction
  - Vivaldi, PIC, GNP, ICS, Virtual Landmarks, PCoord, NPS, Lighthouse, IDMaps
  - Require substantial work to support applications

- Application-specific approaches
  - Closest node [Meridian, Oasis, ...]
  - Detour routing [OneHop shortest path, ...]
  - Clustering [SDIMS, ...]

- Need for a different abstraction
  - As general as coordinates
  - Inherently providing app-specific functionality
Prediction Trees

Latency between nodes is estimated by their path length on the tree

Tree of Virtual Routing Nodes

Interior: Virtual Routing Nodes
Leaves: Physical End-hosts
Tree Embeddings

• Idea: Approximate distances by a tree

• Goal: Map points of metric space \((X,d)\) to nodes of a tree \(T\) such that distances are preserved as well as possible

• Metric: \(d(x,y)\). Tree: \(T(x,y)\). W.l.o.g., assume \(T(x,y) \geq d(x,y)\)

• Distortion of mapping is defined as

\[
\max_{(x,y) \in X} \frac{T(x,y)}{d(x,y)}
\]

• General metrics: Embedding the \(n\)-cycle requires distortion \(\Omega(n)\) [Rabinovich,Raz 98]
• Intuitively: Hierarchical organization $\rightarrow$ tree-like properties
Do Prediction Trees Work?

• The 4-Points Condition:

\[ d(s,u)+d(t,v) = d(s,t)+d(u,v) \geq d(s,u)+d(t,v) \]

Distance metric is tree metric \(\iff\) 4PC is satisfied for every 4 points.
Do Prediction Trees Work?

- **ε-4-Points Condition:**
  - If \( d(s,v) + d(u,t) \geq d(s,t) + d(u,v) \geq d(s,v) + d(t,u) \):
    \[
    d(s,v) + d(u,t) \leq d(s,t) + d(u,v) + 2\varepsilon \cdot \min\{d(s,v), d(t,u)\}
    \]

- Internet latencies are close to a tree metric:
Challenge

• Practical and theoretical challenges:

• How well can the Internet be approximated by a tree?

• How well can relaxed tree-metrics be mapped to a tree?
  – Given: Metric satisfying $\epsilon$-4-points condition
  – Goal: Minimize distortion as a function of $\epsilon$
Main Results

• **Upper Bound:**

  - Algorithm which computes tree with distortion $(1 + \varepsilon)^{c_1 \cdot \log n}$

• **Lower Bound:**

  - Metric satisfying $\varepsilon$-4PC which requires distortion $(1 + \varepsilon)^{c_2 \cdot \log n}$
Lower Bound

- Consider metric space \((X,d)\):

\[ X = \{ x_1, \ldots, x_n \} , \quad d(x_i, x_j) = (1 + \delta)^{\log_2 |j - i|} \]

- Lemma 1: \((X,d)\) satisfies \(\varepsilon\)-4-points condition for \(\varepsilon = \Theta(\delta)\)

- Lemma 2: Every tree embedding of \((X,d)\) has distortion \(\geq (1 + \varepsilon)^{c \cdot \log n}\) for some constant \(c\).
ε–4-Point Condition

• In this talk: Consider slightly simpler metric \((X,d)\):

\[ X = \{ x_1, \ldots, x_n \}, \quad d(x_i, x_j) = 1 + 2\varepsilon \cdot \log_2 |j - i| \]

• Lemma: \((X,d)\) satisfies \(\varepsilon\)-4-points condition.

• Proof sketch:

Given 4 points \(x_a, x_b, x_c, x_d\):

Smallest distance \(\geq 1\)

Largest distance sum \(- 2^{nd}\)-largest distance sum \(\leq 2\cdot\varepsilon\)
Any Embedding of \((X,d)\) has Large Distortion

- Lemma 2: Tree embedding of \((X,d)\) has distortion \(\geq 1 + \frac{2}{3} \cdot \varepsilon \cdot \log n\)

- Proof (by contradiction):
  
  - Assume there is a tree \(T\) with distortion \(< 1 + \frac{2}{3} \cdot \varepsilon \cdot \log n\)
  
  - Denote tree distance by \(T(x,y)\), distance in metric by \(d(x,y)\)
  
  - W.l.o.g., assume that \(T\) is dominating, i.e. \(T(x,y) \geq d(x,y)\)
  
  - Contradiction assumption:
    \[
    \forall x, y \in X : d(x, y) \leq T(x, y) < \left(1 + \frac{2}{3} \varepsilon \log n\right) \cdot d(x, y)
    \]
Switching Indices

• Consider node $r$ separating $T$ into two subtrees $T_1$ and $T_2$

• Index $i$ is switching index if $x_i$ and $x_{i+1}$ are in different subtrees of $r$

• Lemma: There are no switching indices $a$ and $b$ with $b-a \geq n^{1/3}+1$

• Proof:

\[
T(x_a, x_{a+1}) = T(x_a, r) + T(x_{a+1}, r) < 1 + \frac{2}{3} \varepsilon \log n
\]

\[
T(x_b, x_{b+1}) = T(x_b, r) + T(x_{b+1}, r) < 1 + \frac{2}{3} \varepsilon \log n
\]

\[
T(x_a, r) + T(x_{a+1}, r) + T(x_b, r) + T(x_{b+1}, r) < 2 + \frac{4}{3} \varepsilon \log n
\]

\[
T(x_a, r) + T(x_{b+1}, r) \geq T(x_a, x_{b+1}) \geq 1 + \frac{2}{3} \varepsilon \log n
\]

\[
T(x_a, r) + T(x_{b+1}, r) \geq T(x_a, x_{b+1}) \geq 1 + \frac{2}{3} \varepsilon \log n
\]

\[
T(x_a, r) + T(x_{a+1}, r) + T(x_b, r) + T(x_{b+1}, r) \geq 2 + \frac{4}{3} \varepsilon \log n
\]

$x_b \in T_1$, $x_{b+1} \in T_2$: analogous.

PODC 2007
Light and Heavy Subtrees

• All nodes in X are leaves of T and all inner nodes have degree 3
• Light subtrees: subtrees with $< n^{1/3} + 1$ leaves
• Heavy subtrees: all others...
• Lemma: No node has 3 heavy subtrees
• Proof Sketch:
  – Assume that node $r$ has 3 heavy subtrees
  – $a$: smallest index s.t. $x_a, x_{a+1}$ in diff. subtrees
  – $b$: largest index s.t. $x_b, x_{b+1}$ in diff. subtrees
  – $a$ and $b$ are switching indices!
  – E.g. $x_a, x_b \in T_1$, $x_{a+1} \in T_2$, $x_{b+1} \in T_3$
A Long Path...

- Nodes with 2 heavy subtrees lie on a path:

Assume $x$, $y$, $z$ not on a path:

Node $r$ where paths from $x,y,z$ meet

Nodes $x$, $y$, $z$ have 2 heavy subtrees

$\implies r$ has 3 heavy subtrees
A Long Path...

• Nodes with two heavy subtrees lie on a path:

• All subtrees are light (<n^{1/3}+1 leaves)

• Length of path > (1-o(1))\cdot n^{2/3}
A Long Path...

- Partition path into minimal segments with at least $n^{2/3}+1$ leaves:

- Number of segments: $(1-o(1)) \cdot n^{1/3}$
Segments are long

- Consider a single segment

  - $a$: smallest index such that $x_a$ and $x_{a+1}$ are in different subtrees
  - $b$: largest index such that $x_b$ and $x_{b+1}$ are in different subtrees
  - Of $x_a, x_{a+1}, x_b, x_{b+1}$, two are in same subtree and $b-a \geq n^{2/3}+1$
  - Two nodes in $T_1$: $a$ and $b$ switching indices for split at $u$
  - Two nodes in $T_3$ analogous $\Rightarrow$ two nodes are in $T_2$, say $x_{a+1}, x_b$
Segments are long

\[ d(x_a, x_{a+1}) = 1 \implies \min \{T(x_a, u), T(x_{a+1}, u)\} \leq \frac{1}{2} \cdot \left(1 + \frac{2}{3} \varepsilon \log n\right) \]

\[ T(x_a, u) \text{ and } T(x_b, v) \leq \frac{1}{2} \cdot \left(1 + \frac{2}{3} \varepsilon \log n\right) \]

\[ 1 + \frac{2}{3} \varepsilon \log n + T(u, v) \geq T(x_a, u) + T(u, v) + T(v, x_b) \geq T(x_a, x_b) \geq 1 + \frac{4}{3} \varepsilon \log n \]

\[ T(u, v) \geq \frac{2}{3} \varepsilon \log n \]
Contradiction

\[ T(u, v) \geq \frac{2}{3} \varepsilon \log n \]

- \((1-o(1)) \cdot n^{1/3}\) segments: \( T(x, y) \geq 1 + (1 - o(1)) \cdot n^{1/3} \cdot \frac{2}{3} \varepsilon \log n \)

\[
\frac{T(x, y)}{d(x, y)} \geq \frac{1 + (1 - o(1)) \cdot n^{1/3} \cdot \frac{2}{3} \varepsilon \log n}{1 + 2\varepsilon \log n} > 1 + \frac{2}{3} \varepsilon \log n
\]
Summary Results

- **Theorem (lower bound):** Metric spaces satisfying $\varepsilon$-4PC don’t embed into trees with distortion less than $(1 + \varepsilon)^{c \cdot \log n}$.

- **Corollary:** For all $n$ and $\varepsilon$, metric space $(X,d)$ where every 4 points embed into a tree with distortion $1 + \varepsilon$, but where distortion of embedding of $(X,d)$ is $\geq (1 + \varepsilon)^{c \cdot \log n}$.

- **Generalization:** Metric spaces $(X,d)$ where every $k$ points embed into a tree with distortion $1 + \varepsilon$, but where distortion of every embedding of $(X,d)$ is $\geq (1 + \varepsilon)^{c' \cdot \log n / \log k}$.

- **Theorem (upper bound):** Metric spaces satisfying $\varepsilon$-4PC embed into trees with distortion $(1 + \varepsilon)^{d \cdot \log n}$.
Additional Results

- **Additive Distortion:**
  Metric spaces for which every $k$ points embed into a tree with additive distortion $\delta$ embed into trees with additive distortion $\Theta(\delta \log n)$
  
  (main techniques needed already appear in [Gromov 87])

- **Distance Labeling:**
  $\varepsilon$-4PC metric space for which every $(1 + \varepsilon)^{\log k}$-approximate distance labeling needs labels of size $n^{\Omega(1/k)}$
  
  (polylog labels $\xrightarrow{}$ distortion $(1 + \varepsilon)^{\Omega(\log \log n)}$)
  
  (similar to proof for additive distortion in [Gavoille,Ly 05])