# A Summary of TLA$^+$

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6 Jun 2000

## Module-Level Constructs

MODULE $M$

Begins the module or submodule named $M$.

EXTENDS $M_1, \ldots, M_n$

Incorporates the declarations, definitions, assumptions, and theorems from the modules named $M_1, \ldots, M_n$ into the current module.

CONSTANTS $C_1, \ldots, C_n$ (1)

Declares the $C_j$ to be constant parameters (rigid variables). Each $C_j$ is either an identifier or has the form $C(_, \ldots, _)$, the latter form indicating that $C$ is an operator with the indicated number of arguments.

VARIABLES $x_1, \ldots, x_n$ (1)

Declares the $x_j$ to be variables (parameters that are flexible variables).

ASSUME $P$

Asserts $P$ as an assumption.

$F(x_1, \ldots, x_n) \triangleq exp$

Defines $F$ to be the operator such that $F(e_1, \ldots, e_n)$ equals $exp$ with each identifier $x_k$ replaced by $e_k$. (For $n = 0$, it is written $F \triangleq exp$.)

$f[x \in S] \triangleq exp$ (2)

Defines $f$ to be the function with domain $S$ such that $f[x] = exp$ for all $x$ in $S$. (The symbol $f$ may occur in $exp$, allowing a recursive definition.)

(1) The terminal S in the keyword is optional.

(2) $x \in S$ may be replaced by a comma-separated list of items $v \in S$, where $v$ is either a comma-separated list or a tuple of identifiers.

INSTANCE $M$ WITH $p_1 \leftarrow e_1,\ \ldots\ ,\ p_m \leftarrow e_m$

For each defined operator $F$ of module $M$, this defines $F$ to be the operator whose definition is obtained from the definition of $F$ in $M$ by replacing each declared constant or variable $p_j$ of $M$ with $e_j$. (If $m = 0$, the WITH is omitted.)

$N(x_1, \ldots, x_n) \triangleq$ INSTANCE $M$ WITH $p_1 \leftarrow e_1,\ \ldots\ ,\ p_m \leftarrow e_m$

For each defined operator $F$ of module $M$, this defines $N(d_1, \ldots, d_n)!F$ to be the operator whose definition is obtained from the definition of $F$ by replacing each declared constant or variable $p_j$ of $M$ with $e_j$, and then replacing each identifier $x_k$ with $d_k$. (If $m = 0$, the WITH is omitted.)

THEOREM $P$

Asserts that $P$ can be proved from the definitions and assumptions of the current module.

LOCAL *def*

Makes the definition(s) of *def* (which may be a definition or an INSTANCE statement) local to the current module, thereby not obtained when extending or instantiating the module.

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Ends the current module or submodule.

## The Constant Operators

### Logic

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \equiv$

TRUE FALSE BOOLEAN [the set {TRUE, FALSE}]

$\forall\, x : p \qquad \exists\, x : p \qquad \forall\, x \in S : p$ (1) $\qquad \exists\, x \in S : p$ (1)

CHOOSE $x : p$ [An $x$ satisfying $p$] CHOOSE $x \in S : p$ [An $x$ in $S$ satisfying $p$]

### Sets

$= \quad \neq \quad \in \quad \notin \quad \cup \quad \cap \quad \subseteq \quad \setminus$ [set difference]

| | |
|---|---|
| $\{e_1, \ldots, e_n\}$ | [Set consisting of elements $e_i$] |
| $\{x \in S : p\}$ (2) | [Set of elements $x$ in $S$ satisfying $p$] |
| $\{e : x \in S\}$ (1) | [Set of elements $e$ such that $x$ in $S$] |
| SUBSET $S$ | [Set of subsets of $S$] |
| UNION $S$ | [Union of all elements of $S$] |

### Functions

| | |
|---|---|
| $f[e]$ | [Function application] |
| DOMAIN $f$ | [Domain of function $f$] |
| $[x \in S \mapsto e]$ (1) | [Function $f$ such that $f[x] = e$ for $x \in S$] |
| $[S \to T]$ | [Set of functions $f$ with $f[x] \in T$ for $x \in S$] |
| $[f$ EXCEPT $![e_1] = e_2]$ (3) | [Function $\widehat{f}$ equal to $f$ except $\widehat{f}[e_1] = e_2$. An @ in $e_2$ equals $f[e_1]$.] |

### Records

| | |
|---|---|
| $e.h$ | [The $h$-component of record $e$] |
| $[h_1 \mapsto e_1, \ldots, h_n \mapsto e_n]$ | [The record whose $h_i$ component is $e_i$] |
| $[h_1 : S_1, \ldots, h_n : S_n]$ | [Set of all records with $h_i$ component in $S_i$] |
| $[r$ EXCEPT $!.h = e]$ (3) | [Record $\widehat{r}$ equal to $r$ except $\widehat{r}.h = e$. An @ in $e$ equals $r.h$.] |

### Tuples

| | |
|---|---|
| $e[i]$ | [The $i^{\text{th}}$ component of tuple $e$] |
| $\langle e_1, \ldots, e_n \rangle$ | [The $n$-tuple whose $i^{\text{th}}$ component is $e_i$] |
| $S_1 \times \ldots \times S_n$ | [The set of all $n$-tuples with $i^{\text{th}}$ component in $S_i$] |

### Strings and Numbers

| | |
|---|---|
| "$\mathsf{c}_1 \ldots \mathsf{c}_n$" | [A literal string of $n$ characters] |
| STRING | [The set of all strings] |
| $d_1 \ldots d_n \quad d_1 \ldots d_n \,.\, d_{n+1} \ldots d_m$ | [Numbers (where the $d_i$ are digits)] |

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(1) $x \in S$ may be replaced by a comma-separated list of items $v \in S$, where $v$ is either a comma-separated list or a tuple of identifiers.

(2) $x$ may be an identifier or tuple of identifiers.

(3) $![e_1]$ or $!.h$ may be replaced by a comma separated list of items $!\,a_1 \cdots a_n$, where each $a_i$ is $[e_i]$ or $.h_i$ .

**Miscellaneous Constructs**

| | |
|---|---|
| IF $p$ THEN $e_1$ ELSE $e_2$ | [$e_1$ if $p$ true, else $e_2$] |
| CASE $p_1 \rightarrow e_1 \,\Box\, \ldots \,\Box\, p_n \rightarrow e_n$ | [Some $e_i$ such that $p_i$ true] |
| CASE $p_1 \rightarrow e_1 \,\Box\, \ldots \,\Box\, p_n \rightarrow e_n \,\Box$ OTHER $\rightarrow e$ | [Some $e_i$ such that $p_i$ true, or $e$ if all $p_i$ are false] |

LET $d_1 \triangleq e_1 \;\ldots\; d_n \triangleq e_n$ IN $e$ [$e$ in the context of the definitions]

$\bigwedge\limits_{\vdots} p_1 \ldots p_n$ [the conjunction $p_1 \wedge \ldots \wedge p_n$]

$\bigvee\limits_{\vdots} p_1 \ldots p_n$ [the disjunction $p_1 \vee \ldots \vee p_n$]

## The Action Operators

| | |
|---|---|
| $e'$ | [The value of $e$ in the final state of a step] |
| $[A]_e$ | $[A \vee (e' = e)]$ |
| $\langle A \rangle_e$ | $[A \wedge (e' \neq e)]$ |
| ENABLED $A$ | [An $A$ step is possible] |
| UNCHANGED $e$ | $[e' = e]$ |
| $A \cdot B$ | [Composition of actions] |

## The Temporal Operators

| | |
|---|---|
| $\Box F$ | [$F$ is always true] |
| $\Diamond F$ | [$F$ is eventually true] |
| $\mathrm{WF}_e(A)$ | [Weak fairness for action $A$] |
| $\mathrm{SF}_e(A)$ | [Strong fairness for action $A$] |
| $F \rightsquigarrow G$ | [$F$ leads to $G$] |
| $F \stackrel{+}{\triangleright} G$ | [$F$ guarantees $G$ (an assumption/guarantee specification)] |
| $\exists\, x : F$ | [Temporal existential quantification (hiding).] |
| $\forall\, x : F$ | [Temporal universal quantification.] |

## User-Definable Operator Symbols

### Infix Operators

| | | | | | |
|---|---|---|---|---|---|
| $+$ (1) | $-$ (1) | $*$ (1) | $/$ (2) | $\circ$ (3) | $++$ |
| $\div$ (1) | $\%$ (1) | $\hat{}$ (1,4) | $..$ (1) | $\ldots$ | $--$ |
| $\oplus$ (5) | $\ominus$ (5) | $\otimes$ | $\oslash$ | $\odot$ | $**$ |
| $<$ (1) | $>$ (1) | $\leq$ (1) | $\geq$ (1) | $\sqcap$ | $//$ |
| $\prec$ | $\succ$ | $\preceq$ | $\succeq$ | $\sqcup$ | $\hat{}\hat{}$ |
| $\ll$ | $\gg$ | $<:$ | $:>$ (6) | $\&$ | $\&\&$ |
| $\sqsubset$ | $\sqsupset$ | $\sqsubseteq$ (5) | $\sqsupseteq$ | $\mid$ | $\mid\mid$ |
| $\subset$ | $\supset$ | | $\supseteq$ | $\star$ | $\%\%$ |
| $\vdash$ | $\dashv$ | $\models$ | $\doteq$ | $\bullet$ | $\#\#$ |
| $\sim$ | $\simeq$ | $\approx$ | $\cong$ | $\$$ | $\$\$$ |
| $:=$ | $::=$ | $\asymp$ | $\doteq$ | $?$ | $??$ |
| $\propto$ | $\wr$ | $\uplus$ | $\bigcirc$ | $!!$ | $@@$ (6) |

### Postfix Operators (7)

$\hat{}+$ $\hat{}*$ $\hat{}\#$

### Prefix Operator

$-$ (8)

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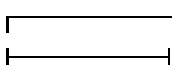
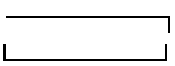
(1) Defined by the *Naturals*, *Integers*, and *Reals* modules.
(2) Defined by the *Reals* module.
(3) Defined by the *Sequences* module.
(4) $x\hat{}y$ is printed as $x^y$.
(5) Defined by the *Bags* module.
(6) Defined by the *TLC* module.
(7) $e\hat{}+$ is printed as $e^+$, and similarly for $\hat{}*$ and $\hat{}\#$.
(8) Defined by the *Integers* and *Reals* modules.

## ASCII Representations of Symbols

| | | | | | |
|---|---|---|---|---|---|
| $\wedge$ | `/\` or `\land` | $\vee$ | `\/` or `\lor` | $\Rightarrow$ | `=>` |
| $\neg$ | `~` or `\lnot` or `\neg` | $\equiv$ | `<=>` or `\equiv` | $\triangleq$ | `==` |
| $\in$ | `\in` | $\neq$ | `#` or `/=` | $'$ | `'` |
| $\langle$ | `<<` | $\rangle$ | `>>` | $\Box$ | `[]` |
| $<$ | `<` | $>$ | `>` | $\Diamond$ | `<>` |
| $\leq$ | `\leq` or `=<` | $\geq$ | `\geq` or `>=` | $\rightsquigarrow$ | `~>` |
| $\ll$ | `\ll` | $\gg$ | `\gg` | $\stackrel{+}{\rightarrow}$ | `-+->` |
| $\prec$ | `\prec` | $\succ$ | `\succ` | $\mapsto$ | `\|->` |
| $\preceq$ | `\preceq` | $\succeq$ | `\succeq` | $\div$ | `\div` |
| $\subseteq$ | `\subseteq` | $\supseteq$ | `\supseteq` | $\wr$ | `\wr` |
| $\subset$ | `\subset` | $\supset$ | `\supset` | $\bullet$ | `\bullet` |
| $\sqsubset$ | `\sqsubset` | $\sqsupset$ | `\sqsupset` | $\star$ | `\star` |
| $\sqsubseteq$ | `\sqsubseteq` | $\sqsupseteq$ | `\sqsupseteq` | $\sim$ | `\sim` |
| $\vdash$ | `\|-` | $\dashv$ | `-\|` | $\simeq$ | `\simeq` |
| $\models$ | `\|=` | $=\!\!\mid$ | `=\|` | $\asymp$ | `\asymp` |
| $\rightarrow$ | `->` | $\leftarrow$ | `<-` | $\approx$ | `\approx` |
| $\cap$ | `\cap` or `\intersect` | $\cup$ | `\cup` or `\union` | $\cong$ | `\cong` |
| $\sqcap$ | `\sqcap` | $\sqcup$ | `\sqcup` | $\doteq$ | `\doteq` |
| $\oplus$ | `(+)` or `\oplus` | $\uplus$ | `\uplus` | $\propto$ | `\propto` |
| $\ominus$ | `(-)` or `\ominus` | $\times$ | `\X` or `\times` | $x^y$ | `x^y` (2) |
| $\odot$ | `(.)` or `\odot` | $\cdot$ | `\cdot` | $x^+$ | `x^+` (2) |
| $\otimes$ | `(\X)` or `\otimes` | $\circ$ | `\o` or `\circ` | $x^*$ | `x^*` (2) |
| $\oslash$ | `(/)` or `\oslash` | “s” | `"s"` (1) | $x^\#$ | `x^#` (2) |
| $\exists$ | `\E` | $\forall$ | `\A` | | |
| $\mathbf{\exists}$ | `\EE` | $\mathbf{\forall}$ | `\AA` | | |




(1) $s$ is a sequence of characters. See Section ?? on page ??.
(2) $x$ and $y$ are any expressions.
(3) a sequence of four or more `-` or `=` characters.

## The Most Common Standard Modules

**Modules** *Naturals*, *Integers*, *Reals*

**Define** $+$ $-$ $*$ $/$ $\hat{}$ $..$ $Nat$ $Real$
$\div$ $\%$ $\leq$ $\geq$ $<$ $>$ $Int$ $Infinity$

Prefix $-$ is not defined in *Naturals*.

$a\hat{}b$ denotes $a^b$.

*Nat*, *Int*, and *Real* are the sets of naturals, integers, and real numbers.

$a \,..\, b$ equals $\{n \in Int : a \leq n \leq b\}$.

$a \,\%\, b$ equals $a$ mod $b$, defined so $0 \leq a \,\%\, b < b$, if $b$ is a positive integer.

$\div$ is defined so $a = b * (a \div b) + (a \,\%\, b)$ for $a$ and $b$ integers with $b > 0$.

/ (division) is defined only in *Reals*.

*Infinity* is defined in *Reals* so $-Infinity < r < Infinity$ for all $r \in Real$.

**Module** *Sequences*

**Defines** $\circ$ $Head$ $SelectSeq$ $SubSeq$
*Append* *Len* *Seq* $Tail$

The tuple/sequence $\langle e_1, \ldots, e_n \rangle$ equals the function $[i \in 1 \,..\, n \mapsto e_i]$.

$s \circ t$ is the concatenation of sequences $s$ and $t$.

$Append(\langle e_1, \ldots, e_n \rangle, e_{n+1}) = \langle e_1, \ldots, e_{n+1} \rangle$

$Head(\langle e_1, \ldots, e_n \rangle) = e_1$

$Tail(\langle e_1, \ldots, e_n \rangle) = \langle e_2, \ldots, e_n \rangle$

$Len(\langle e_1, \ldots, e_n \rangle) = n$

$Seq(S)$ is the set of all finite sequences of elements of $S$.

$SubSeq(\langle e_1, \ldots, e_n \rangle, j, k) = \langle e_j, \ldots, e_k \rangle$

$SelectSeq(s, Test)$ is the subsequence of elements $e$ of $s$ satisfying $Test(e)$.

**Module** *FiniteSets*

**Defines** *IsFiniteSet* *Cardinality*

$IsFiniteSet(S)$ is true iff $S$ is a finite set.

$Cardinality(S)$ is the number of elements in $S$, if $S$ is a finite set.