A Near-Optimal Distributed Fully Dynamic Algorithm for Maintaining Sparse Spanners

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The Message-Passing Model

- n processors reside in vertices of an unweighted undirected graph $G = (V, E)$.
  Each processor has a unique id.

- Interconnected via links of E.

- *Short* messages ($O(\log n)$ bits).

- Unlimited computational power.
  Local computation requires zero time.

The Message-Passing Model (Cont.)

Synchronous setting (for this talk).

- Communication in *discrete* rounds.
- Messages sent in the beginning of a round R, arrive before the round $R+1$ starts.

Running Time = #rounds.

Message Complexity = # messages.

Dynamic Model

Edges and vertices may
appear or crash at will.

Motivation for the dynamic model:
real-life networks,
modern ad-hoc, sensor, wireless networks.

Require *simple* algorithms!

- Endpoints of a crashing edge are notified by a link-level protocol.

- Message is lost only if its edge crashes.

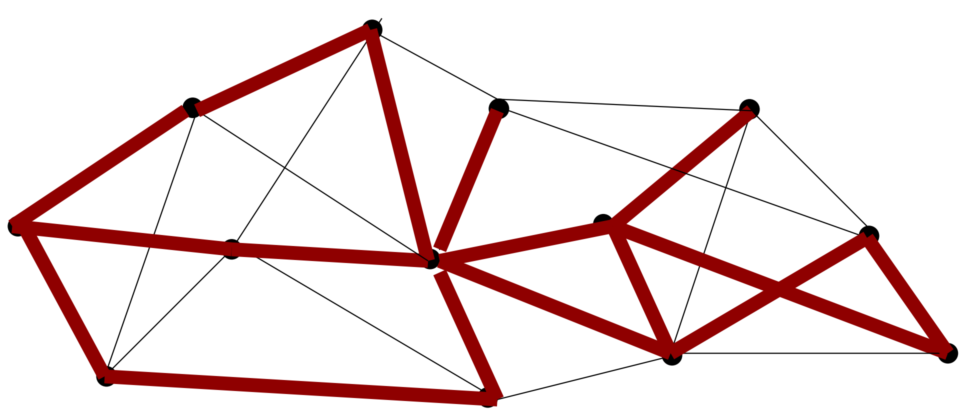
Quiescence Complexity; Spanners

Topology updates cease occuring at time α.
β is the time when all vertices stop processing updates. At this point the algorithm maintains a correct structure.

Quiescence time = $\max\{\beta - \alpha\}$.
Quiescence message = # messages sent within $[\alpha, \beta]$.

$G' = (V, H)$ is a t-*spanner* of $G = (V, E)$, $H \subseteq E$, if $\forall\ u, w \in V$,

$$dist_{G'}(u,w) \ \leq \ t \cdot dist_{G}(u,w) \ .$$



Applications of Spanners

Underlying construct for many distributed algorithms.

- Synchronization.
  [Peleg,Ullman,89],
  [Awerbuch,Peleg,90]

- Routing.
  [Hassin,Peleg,99]

- Approximate Distances and Shortest Paths Computation.
  [Awerbuch,Berger,Cowen,Peleg,93],
  [Elkin,01]

- Broadcast.
  [Awerbuch,Goldreich,Peleg,Vainish,89],
  [Awerbuch,Baratz,Peleg,92]

Distributed Spanners

State-of-the-art distributed *static* algorithm.

[Baswana,Sen,03],
[Baswana,Kavitha,Mehlhorn,Pettie,05]

For $t = 1, 2, \ldots$, and n-vertex G,
constructs $(2t-1)$-spanner with
expected $O(t \cdot n^{1+1/t})$ edges.

Time: $O(t)$.
Message: $O(|E| \cdot t)$.
Space: $O(deg(v) \cdot \log n)$.

Near-optimal tradeoff.

Dynamic State-of-the-Art

[Baswana,Sen] composed with
the simulation technique of
[Awerbuch,Patt-Shamir,Peleg,Saks,92]:
$(2t-1)$-spanner of expected size $O(t \cdot n^{1+1/t})$,
Quiescence time: $O(t \cdot \log^3 n)$.
Quiescence message: $O(t \cdot |E| \cdot \log^3 n)$.
Space: $O(deg(v) \cdot \log^4 n)$.

Drawbacks of APSPS simulation technique:

Extremely *complex* (a reset procedure,
neighborhood covers, a bootstrap technique,
a local rollback).

Heavy local computations - unsuitable
for *simple* devices.

Our Result

$(2t-1)$-spanner of expected size $O(t \cdot n^{1+1/t})$.

Quiescence time: $3t$ instead of $O(t \cdot \log^3 n)$.
Note: $t \leq \log n$.

Quiescence message: worst-case $O(|E| \cdot t)$, expected $O(|E|)$.
Space: $O(deg(v) \cdot \log n)$.
Expected local processing per edge: $O(1)$.

Lower bound: $2t/3$.
$t-1$ under Erdos girth conjecture.

Better performance in purely incremental and purely decremental settings.

In both algorithms: non-adaptive adversary, oblivious to coin tosses.

Memoryless Dynamic Algorithm

Standard approach: maintain *history* of communication, undo operations based on the history.

Very expensive in terms of *local computation*. *Unfeasible* in wireless, sensor, ad-hoc networks.

Our approach: No history stored!

Look for a "replacement" for crashing edges.

Undo operations, but the list-to-undo is deduced from the current state of affairs.

Reminiscent of *online* algorithms.

The Incremental Variant: Initialization

For this talk: only incremental algorithm.

Set a parameter $p \approx n^{-1/t}$.

Each v picks a radius $r(v)$ from the truncated geometric distribution

$\mathbb{P}(r = k) = p^k \cdot (1-p)$, for $k \in [0, \ldots, t-2]$, and $\mathbb{P}(r = t-1) = p^{t-1}$.

Memoryless distribution

$\mathbb{P}(r \geq k+1 \mid r \geq k) = p$ for $k \in [0, 1, \ldots, t-2]$.

[Linial,Saks,92],[Bartal,96]

Labels

Each v has a unique id $I(v)$,
and a label $P(v)$.

Initially, $P(v) \leftarrow I(v)$ $(P(v) \leftarrow (I(v), 0))$.

$P = (B(P), L(P))$, or $P = n \cdot L(P) + B(P)$.

Implicitly, the algorithm maintains a tree cover.

$B(P)$ - the id of a tree τ to which the vertex v labeled by P currently belongs.

$L(P)$ - the distance between v and the root of τ.

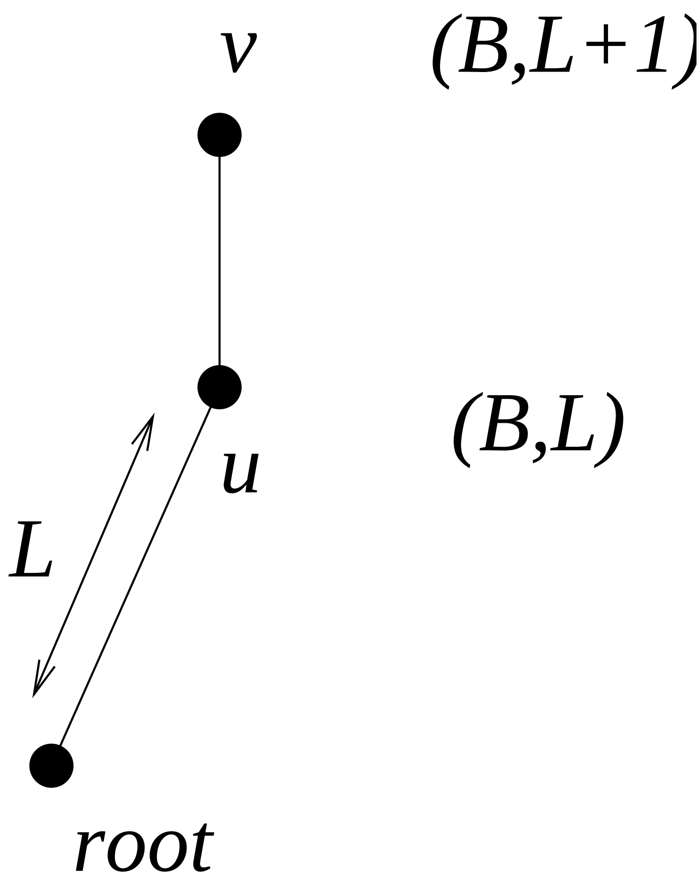
The vertex $w = w_P$ s.t. $I(w) = B(P)$ is the *base* vertex of P.
w_P is the root of the tree $B(P)$.

$r(w_P)$ - maximum distance to which
$B(P) = I(w_P)$ is allowed to propagate.
The tree $B(P)$ cannot be deeper than $r(w_P)$.

$\Rightarrow$ For each label P, $L(P) \leq r(w_P)$.

A label P is *selected* if $L(P) < r(w_P)$.
In this case v may be
an internal vertex of the tree $B(P)$.

Vertices *adopt* labels from their neighbors.
When v adopts a label from u it becomes its child in the tree $B(P)$, $P = P(u)$.
When a label P is adopted, $L(P)$ is incremented, but $B(P)$ stays unchanged.



Data Structures

Every v maintains an edge set $Sp(v)$.

Initially, $Sp(v) = \emptyset$.

$Sp(v)$ grows monotonely.

$Sp(v) = T(v) \cup X(v)$.

$T(v)$ - the *tree edges* of v.

$X(v)$ - the *cross edges* of v.

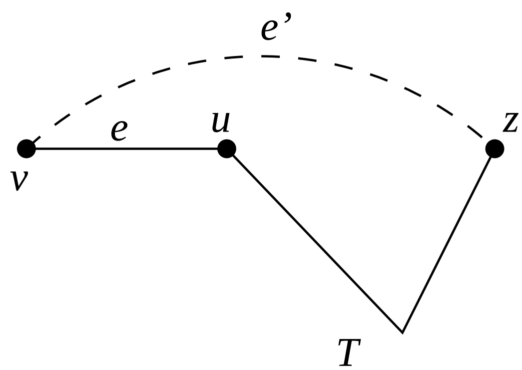
An implicit construction of a *tree cover*. Edges of the tree cover are stored in $T(v)$'s.

The spanner also has edges connecting different trees. Those are edges of $X(v)$'s.

Data Structures (Cont.)

For each vertex v, the algorithm maintains a table $M(v)$.
Initially, $M(v) = \emptyset$.

$M(v)$ is the set of trees to which v is already connected in the spanner.



e'= (v,z) in X(v) ==> B(P(z) in M(v)
B(P(z)) = B(P(u))

$$\Downarrow$$

e can be dropped!

The Algorithm

For $2t$ rounds from the beginning *or* after detecting a new edge do

Go over all received messages and do
while $\exists$ message $P(u)$ with $P(u) \succ P(v)$
    if u is selected
        $B(P(v)) \leftarrow B(P(u))$;
        $L(P(v)) \leftarrow L(P(u)) + 1$;
        $Sp(v) \leftarrow Sp(v) \cup \{e\}$ // add e to $T(v)$
    else if $B(P(u)) \notin M(v)$
        $M(v) \leftarrow M(v) \cup \{B(P(u))\}$;
        $Sp(v) \leftarrow Sp(v) \cup \{e\}$ // add e to $X(v)$
    end-if
end-while
Send to all neighbors the message $P(v)$.

The Algorithm: Discussion

Very simple:

1. One type of messages.

2. The same behavior on each round.

3. A handful of local variables.

4. A basic data structure.

Summary

- Optimal solution for the dynamic distributed spanner problem.

- Memoryless paradigm for devising dynamic distributed algorithms.

- Lower bound of $\Omega(t)$.

- Applications for streaming and dynamic centralized models.

Open Questions

- Applications for the dynamic spanners.
  Synchronization (?), Routing (?),
  Online load balancing (?).

- Applications for the memoryless paradigm.

- Achieve spanner size of $O(n^{1+1/t})$
  instead of $O(t \cdot n^{1+1/t})$.

- Derandomize.
  Less challenging - devise algorithm
  for an adaptive adversary.