Reconstructing Approximate Tree Metrics

Sequoia

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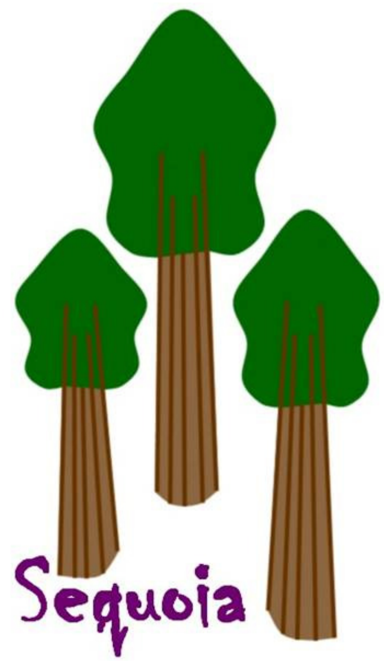
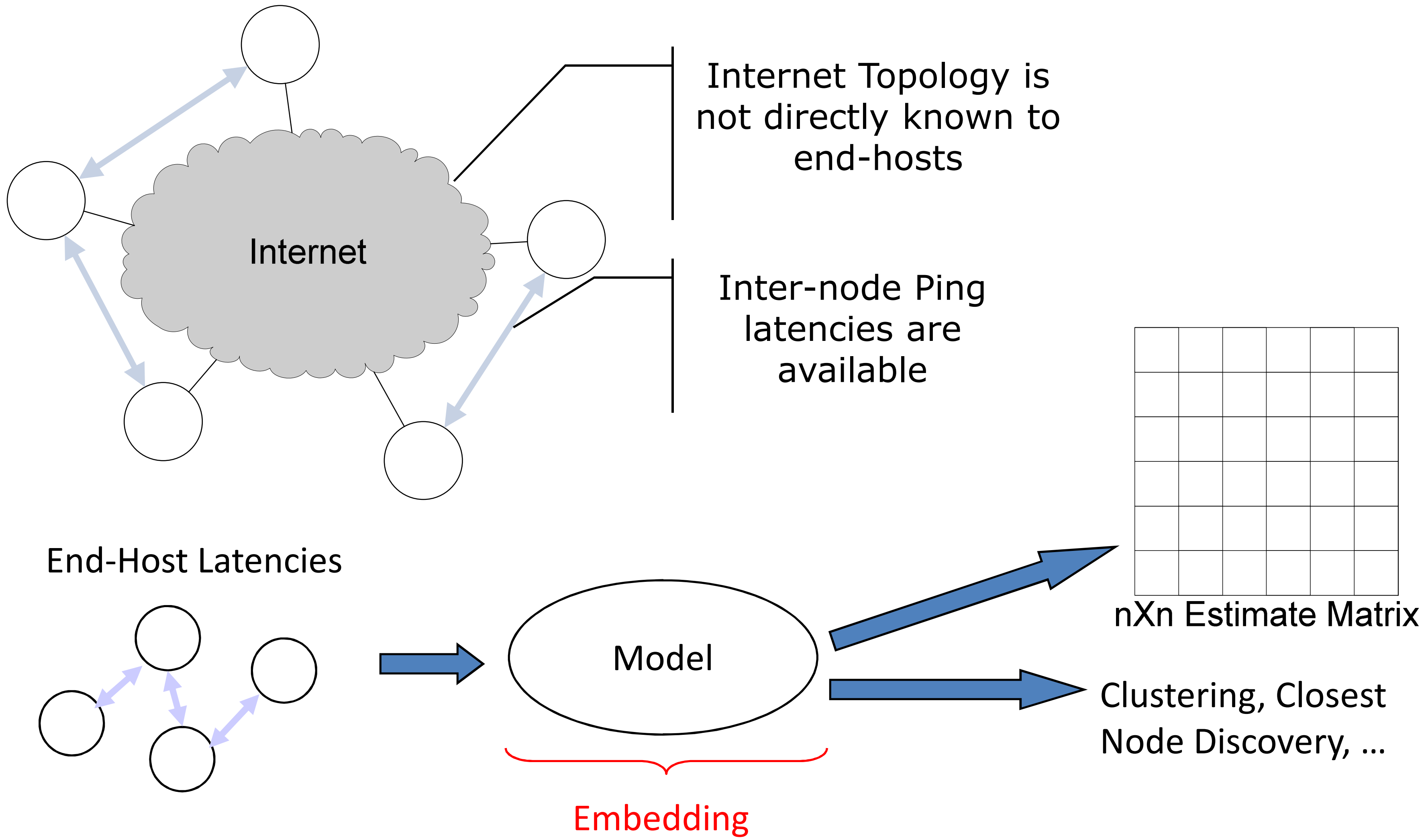
Introduction

- Sequoia: Project at MSR Silicon Valley
- Goals: Distributed system to predict network latencies and provide latency-related services by mapping the network nodes to a tree.
- Latency-enabled functionality:
  - Latency prediction
  - Closest node discovery
  - Locality-based clustering
  - Detour routing
  - etc…



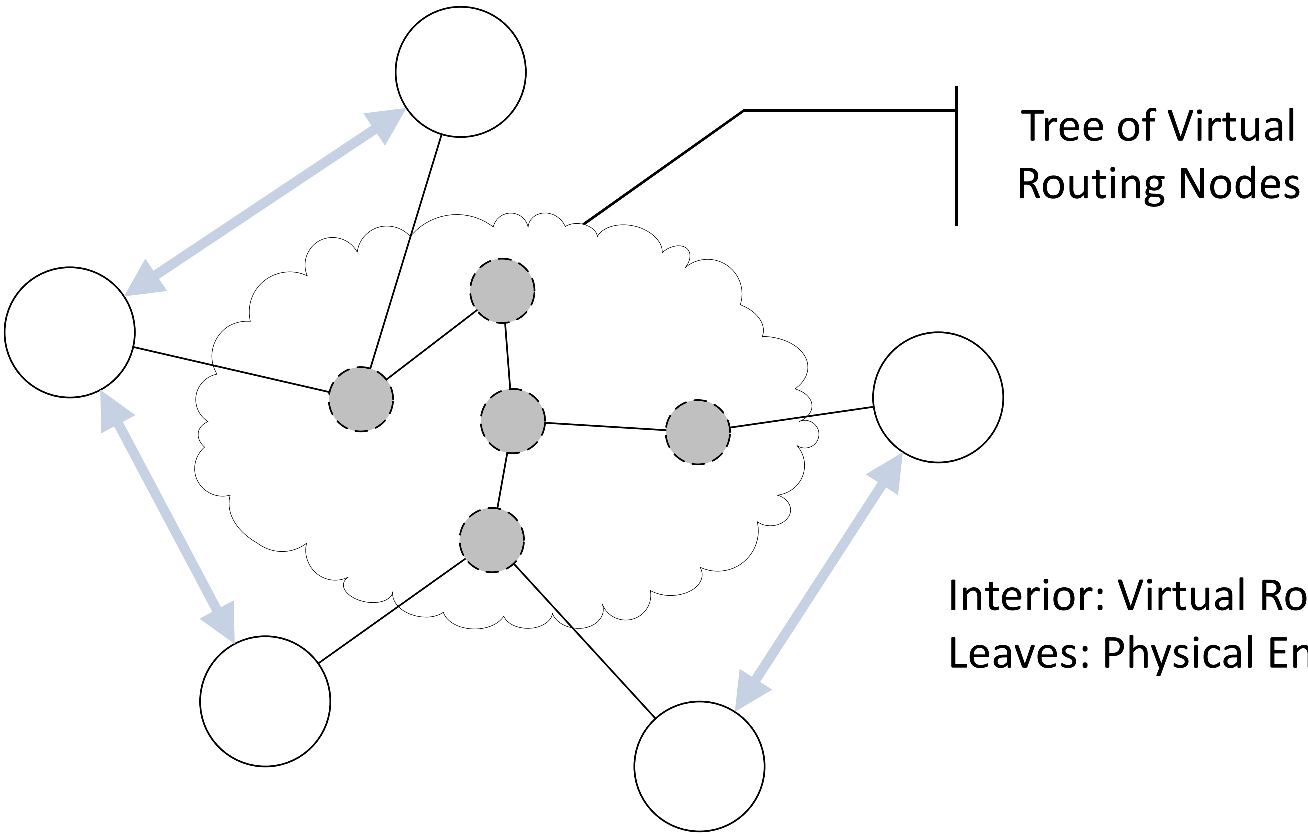
Goals


Internet
Internet Topology is not directly known to end-hosts
Inter-node Ping latencies are available
End-Host Latencies
Model
Embedding
nXn Estimate Matrix
Clustering, Closest Node Discovery, …

Current State of the Art

- Coordinate-based latency prediction
  - Vivaldi, PIC, GNP, ICS, Virtual Landmarks, PCoord, NPS, Lighthouse, IDMaps
  - Require substantial work to support applications
- Application-specific approaches
  - Closest node [Meridian, Oasis, …]
  - Detour routing [OneHop shortest path, …]
  - Clustering [SDIMS, …]
- Need for a different abstraction
  - As general as coordinates
  - Inherently providing app-specific functionality

Prediction Trees



Interior: Virtual Routing Nodes
Leaves: Physical End-hosts

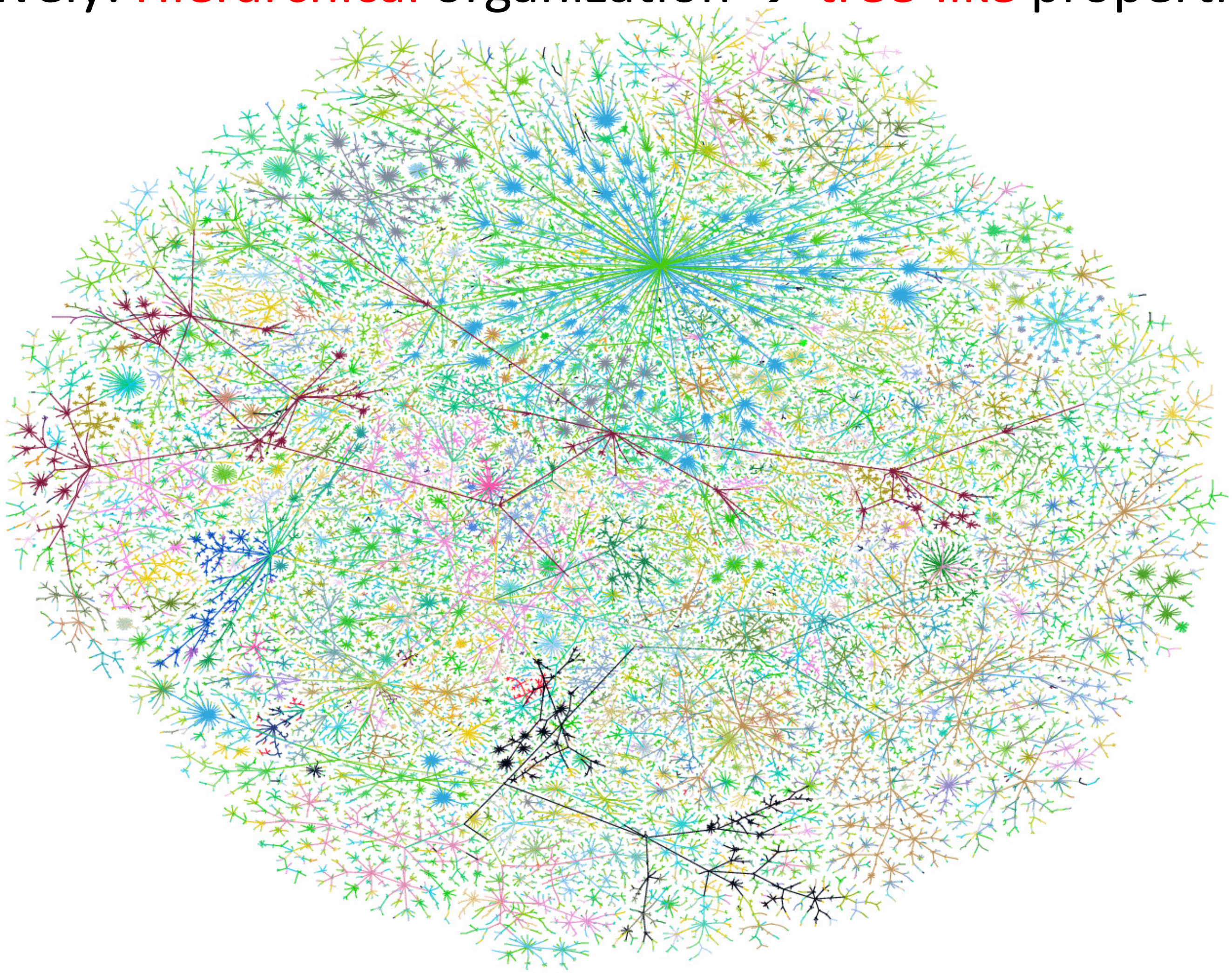
Latency between nodes is estimated by their path length on the tree

Tree Embeddings

- Idea: Approximate distances by a tree
- Goal: Map points of metric space (X,d) to nodes of a tree T such that distances are preserved as well as possible
- Metric: d(x,y). Tree: T(x,y). W.l.o.g., assume $T(x,y) \geq d(x,y)$
- Distortion of mapping is defined as $\max_{(x,y)\in X} \frac{T(x,y)}{d(x,y)}$
- General metrics: Embedding the n-cycle requires distortion $\Omega(n)$ [Rabinovich,Raz 98]

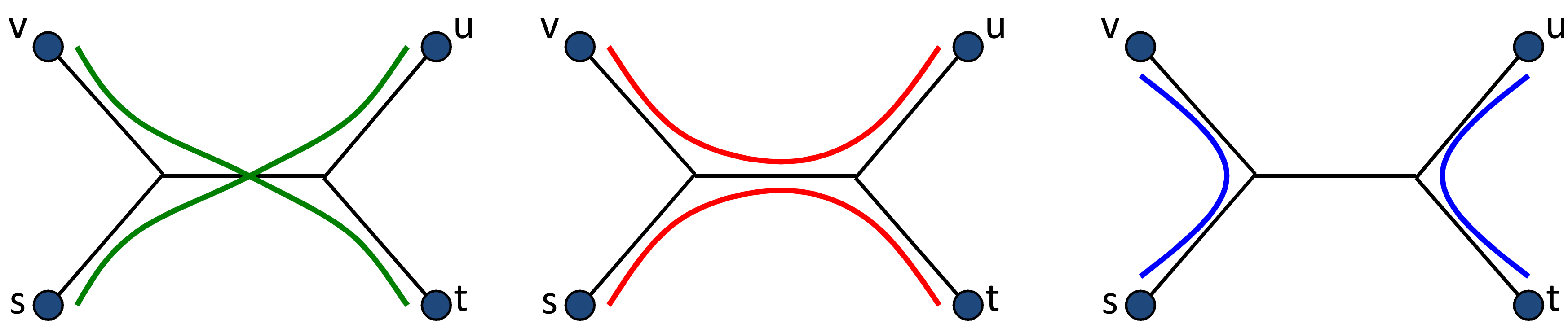
Do Prediction Trees Work?

- Intuitively: Hierarchical organization → tree-like properties



Do Prediction Trees Work?

- **The 4-Points Condition:**



d(s,u)+d(t,v) = d(s,t)+d(u,v) $\geq$ d(s,u)+d(t,v)

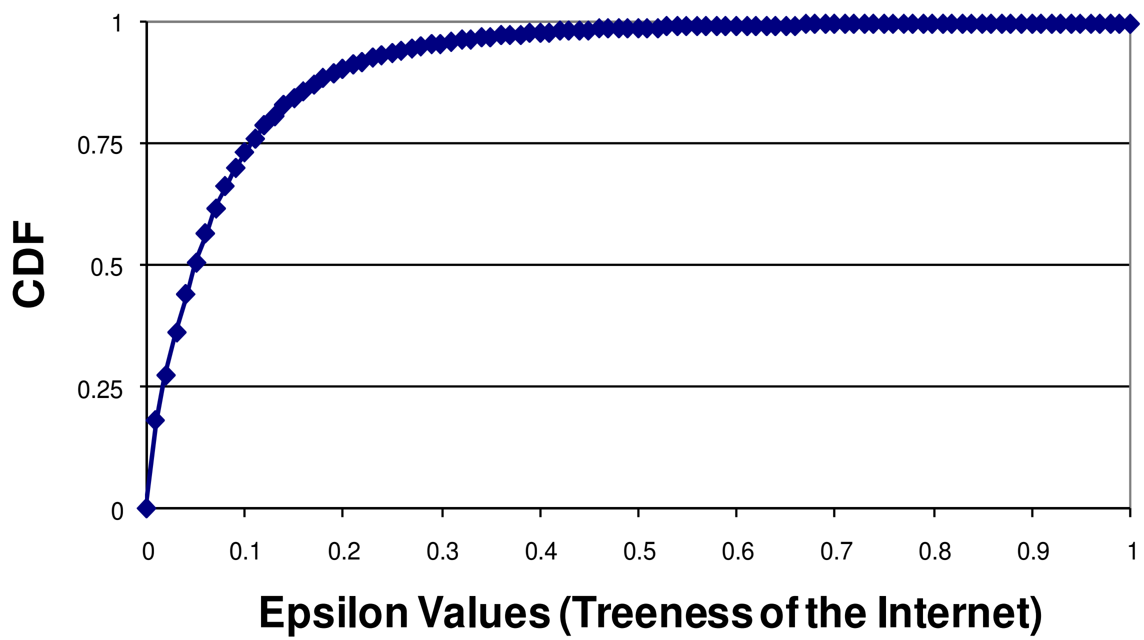
distance metric is tree metric ↔ 4PC is satisfied for every 4 points

Do Prediction Trees Work?

- **ε-4-Points Condition:**
  – If $d(s,v)+d(u,t) \geq d(s,t)+d(u,v) \geq d(s,v)+d(t,u)$:

    $d(s,v)+d(u,t) \leq d(s,t)+d(u,v) + 2\varepsilon \cdot \min\{d(s,v),d(t,u)\}$

- Internet latencies are close to a tree metric:



Challenge

- Practical and theoretical challenges:
- How well can the Internet be approximated by a tree?
- How well can relaxed tree-metrics be mapped to a tree?
  - Given: Metric satisfying ε-4-points condition
  - Goal: Minimize distortion as a function of ε

Main Results

- **Upper Bound:**
- Algorithm which computes tree with distortion $(1+\varepsilon)^{c_1 \cdot \log n}$

- **Lower Bound:**
- Metric satisfying ε-4PC which requires distortion $(1+\varepsilon)^{c_2 \cdot \log n}$

Lower Bound

- Consider metric space (X,d):

$$X = \{x_1, \ldots, x_n\}, \quad d(x_i, x_j) = (1+\delta)^{\log_2 |j-i|}$$

- Lemma 1: (X,d) satisfies ε-4-points condition for ε=Θ(δ)

- Lemma 2: Every tree embedding of (X,d) has distortion $\geq (1+\varepsilon)^{c \cdot \log n}$ for some constant c.

ε–4-Point Condition

- In this talk: Consider slightly simpler metric (X,d):

$$X = \{x_1, \ldots, x_n\}\,, \quad d(x_i, x_j) = 1 + 2\varepsilon \cdot \log_2 |j - i|$$

- Lemma: (X,d) satisfies ε-4-points condition.

- Proof sketch:

Given 4 points x_a, x_b, x_c, x_d:

Smallest distance $\geq$ 1

Largest distance sum – 2nd-largest distance sum $\leq$ 2·ε

Any Embedding of (X,d) has Large Distortion

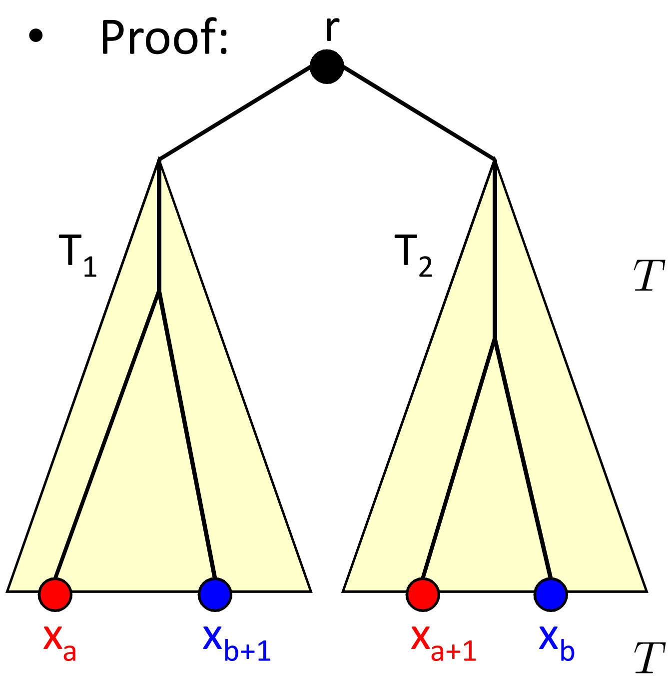
- Lemma 2: Tree embedding of (X,d) has distortion $\geq 1 + \frac{2}{3} \cdot \varepsilon \cdot \log n$

- Proof (by contradiction):
  - Assume there is a tree T with distortion $< 1 + \frac{2}{3} \cdot \varepsilon \cdot \log n$
  - Denote tree distance by T(x,y), distance in metric by d(x,y)
  - W.l.o.g., assume that T is dominating, i.e. $T(x,y) \geq d(x,y)$
  - Contradiction assumption:

$$\forall x, y \in X : d(x,y) \leq T(x,y) < \left(1 + \frac{2}{3}\varepsilon \log n\right) \cdot d(x,y)$$

Switching Indices

- Consider node r separating T into two subtrees T_1 and T_2
- Index i is switching index if x_i and x_{i+1} are in different subtrees of r
- Lemma: There are no switching indices a and b with b-a $\geq n^{1/3}+1$
- Proof:



$$T(x_a, x_{a+1}) = T(x_a, r) + T(x_{a+1}, r) < 1 + \frac{2}{3}\varepsilon \log n$$

$$T(x_b, x_{b+1}) = T(x_b, r) + T(x_{b+1}, r) < 1 + \frac{2}{3}\varepsilon \log n$$

$$T(x_a, r) + T(x_{a+1}, r) + T(x_b, r) + T(x_{b+1}, r) < 2 + \frac{4}{3}\varepsilon \log n$$

$$T(x_a, r) + T(x_{b+1}, r) \geq T(x_a, x_{b+1}) \geq 1 + \frac{2}{3}\varepsilon \log n$$

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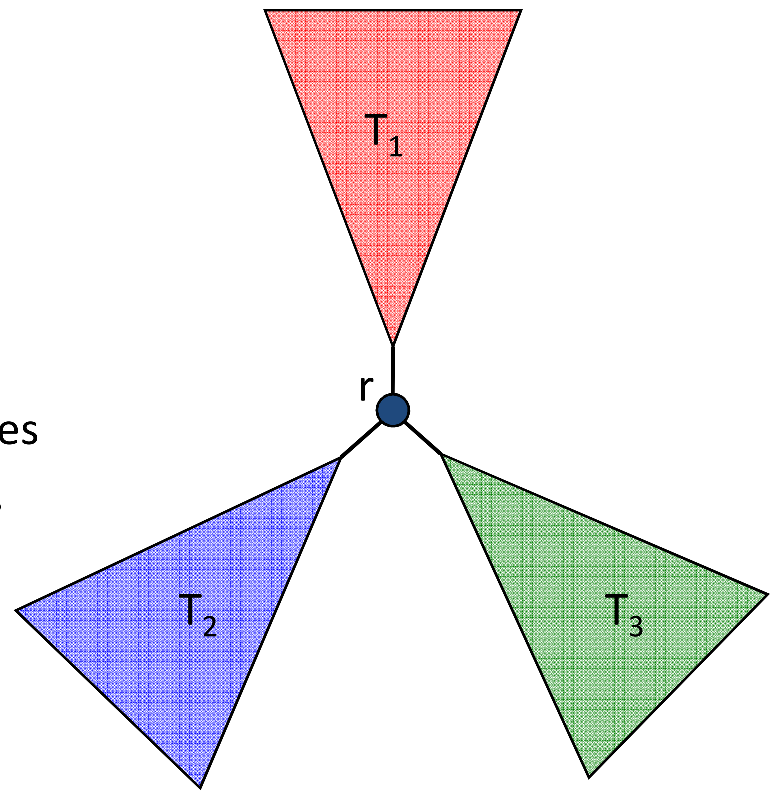
$$T(x_a, r) + T(x_{a+1}, r) + T(x_b, r) + T(x_{b+1}, r) \geq 2 + \frac{4}{3}\varepsilon \log n$$

$x_b \in T_1$, $x_{b+1} \in T_2$: analogous.

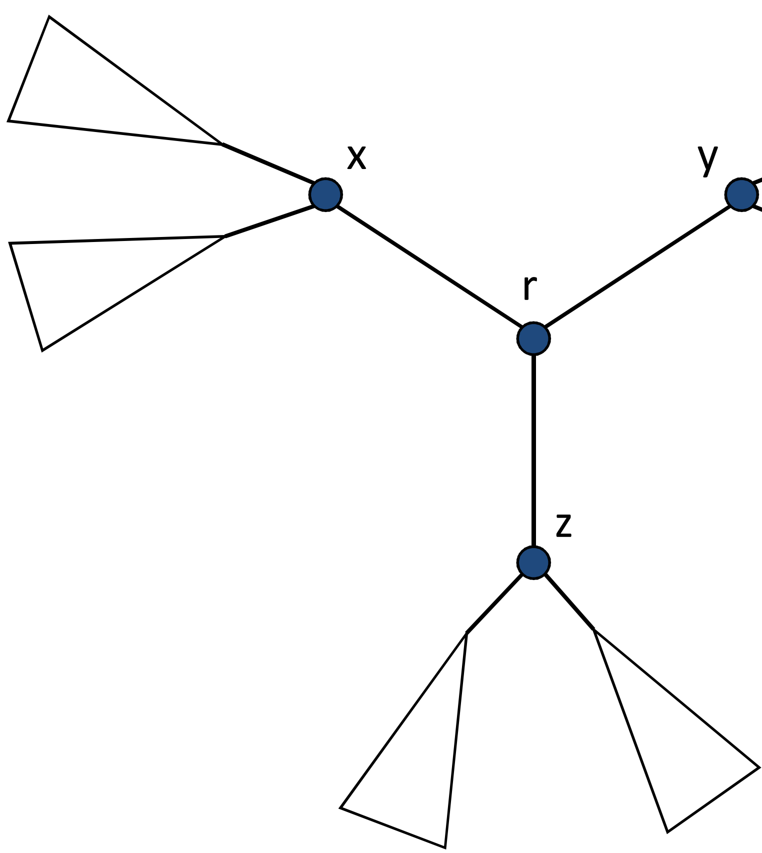
Light and Heavy Subtrees

- All nodes in X are leaves of T and all inner nodes have degree 3
- Light subtrees: subtrees with < $n^{1/3}$+1 leaves
- Heavy subtrees: all others…
- Lemma: No node has 3 heavy subtrees
- Proof Sketch:
  - Assume that node r has 3 heavy subtrees
  - a: smallest index s.t. x_a, x_{a+1} in diff. subtrees
  - b: largest index s.t. x_b,x_{x+1} in diff. subtrees
  - a and b are switching indices!
  - E.g. x_a,$x_b \in T_1$, $x_{a+1} \in T_2$, $x_{b+1} \in T_3$



A Long Path…

- Nodes with 2 heavy subtrees lie on a path:



Assume x, y, z not on a path:

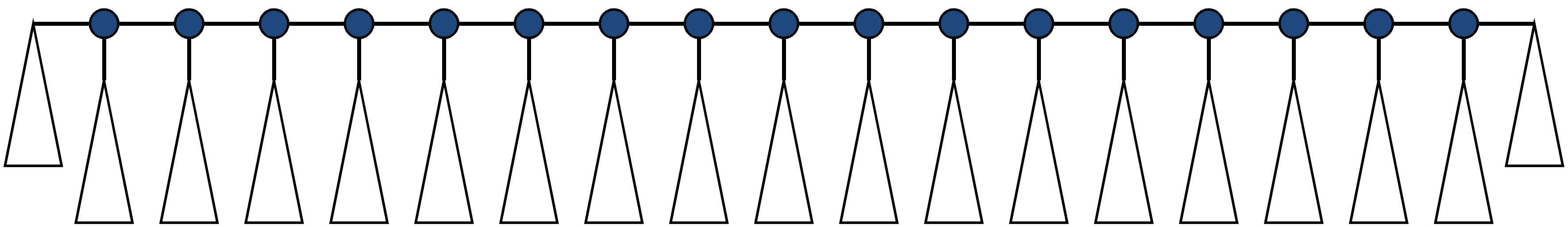
Node r where paths from x,y,z meet

Nodes x, y, z have 2 heavy subtrees

→ r has 3 heavy subtrees

A Long Path…

- Nodes with two heavy subtrees lie on a path:

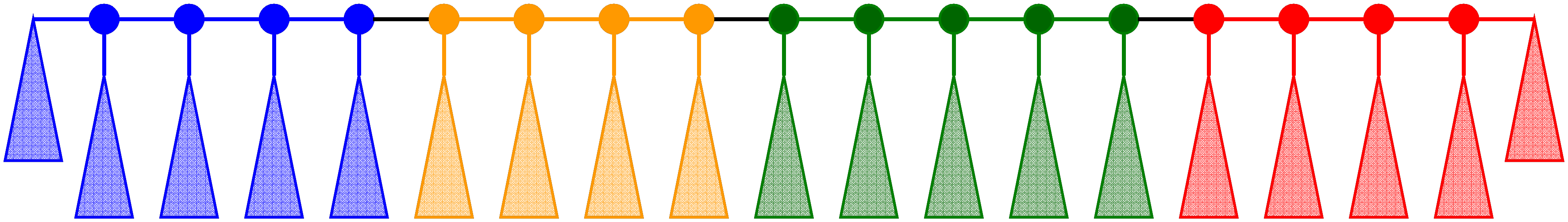


- All subtrees are light (<$n^{1/3}$+1 leaves)

- Length of path > $(1-o(1))\cdot n^{2/3}$

A Long Path…

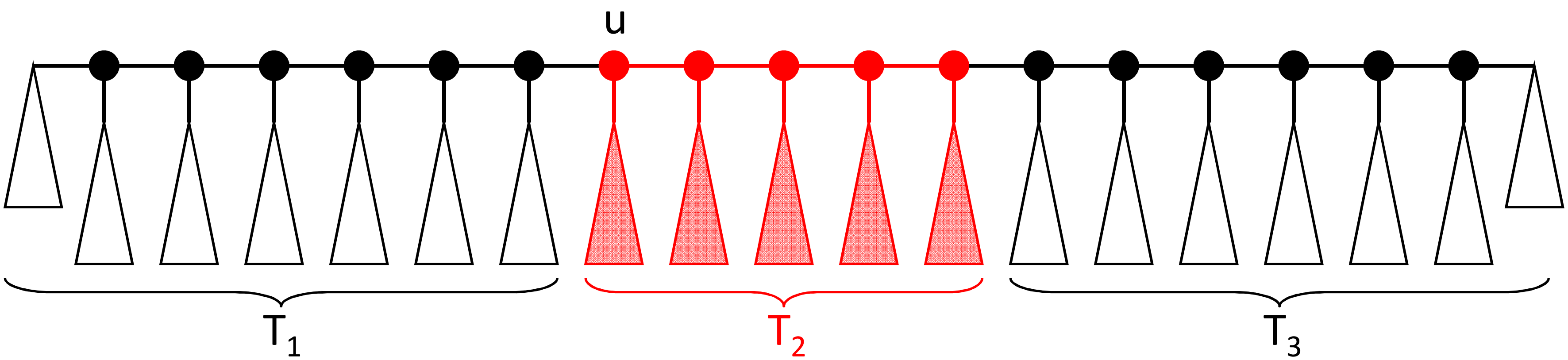
- Partition path into minimal segments with at least $n^{2/3}+1$ leaves:



- Number of segments: $(1-o(1))\cdot n^{1/3}$

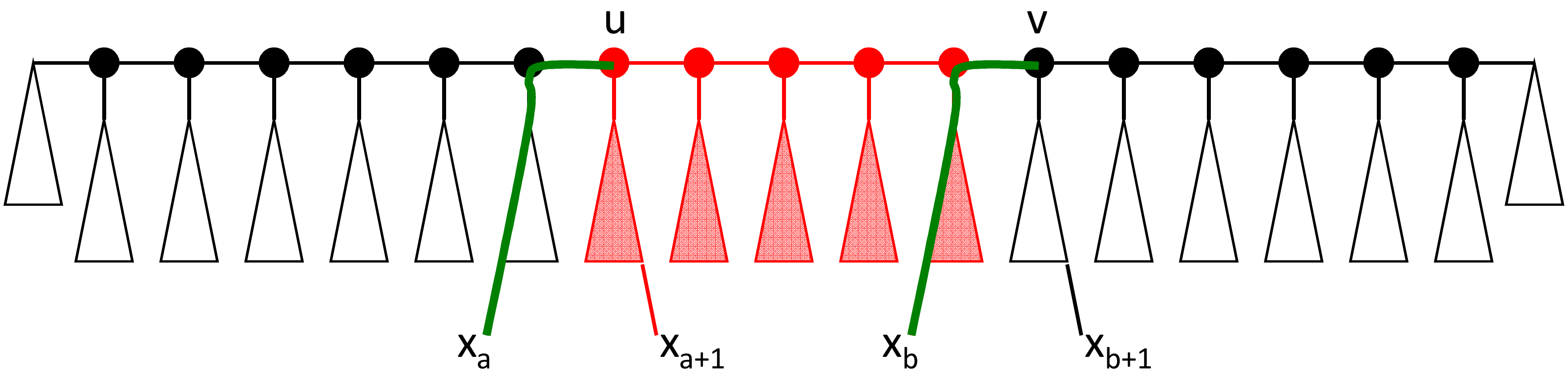
Segments are long

- Consider a single segment



- a: smallest index such that x_a and x_{a+1} are in different subtrees
- b: largest index such that x_b and x_{b+1} are in different subtrees
- Of x_a, x_{a+1}, x_b, x_{b+1}, two are in same subtree and $b-a \geq n^{2/3}+1$
- Two nodes in T_1: a and b switching indices for split at u
- Two nodes in T_3 analogous → two nodes are in T_2, say x_{a+1}, x_b

Segments are long



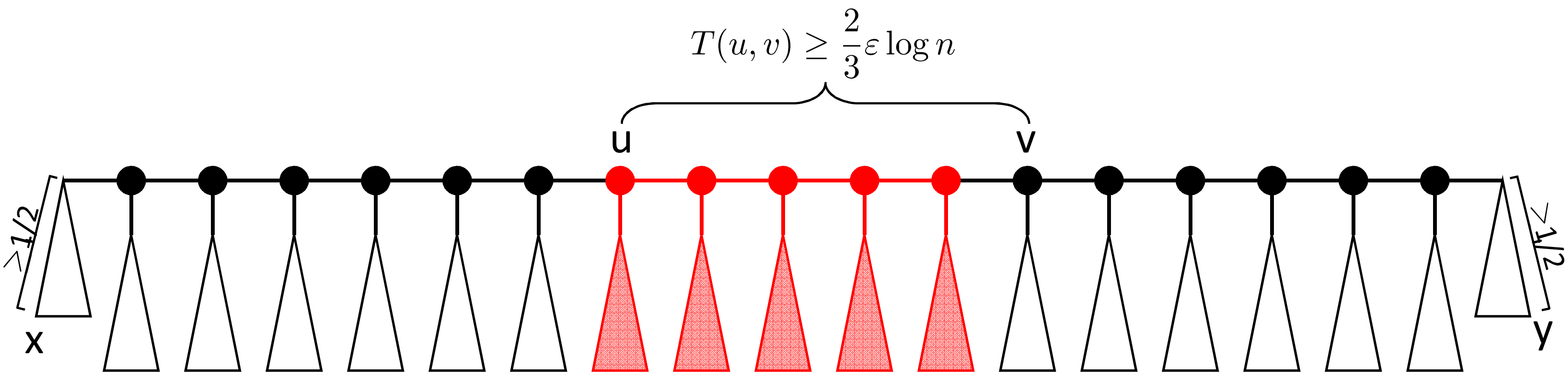
$$d(x_a, x_{a+1}) = 1 \quad \Longrightarrow \quad \min\left\{T(x_a,u), T(x_{a+1},u)\right\} \le \frac{1}{2}\cdot\left(1+\frac{2}{3}\varepsilon\log n\right)$$

$$T(x_a,u) \text{ and } T(x_b,v) \le \frac{1}{2}\cdot\left(1+\frac{2}{3}\varepsilon\log n\right)$$

$$1+\frac{2}{3}\varepsilon\log n + T(u,v) \ge T(x_a,u)+T(u,v)+T(v,x_b) \ge T(x_a,x_b) \ge 1+\frac{4}{3}\varepsilon\log n$$

$$T(u,v) \ge \frac{2}{3}\varepsilon\log n$$

Contradiction



- (1-o(1))·n$^{1/3}$ segments: $T(x,y) \geq 1 + (1 - \mathrm{o}(1)) \cdot n^{1/3} \cdot \frac{2}{3}\varepsilon \log n$

$$\frac{T(x,y)}{d(x,y)} \geq \frac{1 + (1 - \mathrm{o}(1)) \cdot n^{1/3} \cdot \frac{2}{3}\varepsilon \log n}{1 + 2\varepsilon \log n} > 1 + \frac{2}{3}\varepsilon \log n$$

Summary Results

- **Theorem (lower bound)**: Metric spaces satisfying ε-4PC don't embed into trees with distortion less than $(1+\varepsilon)^{c \cdot \log n}$
- **Corollary**: For all n and ε, metric space (X,d) where every 4 points embed into a tree with distortion 1+ε, but where distortion of embedding of (X,d) is $\geq (1+\varepsilon)^{c \cdot \log n}$
- **Generalization**: Metric spaces (X,d) where every k points embed into a tree with distortion 1+ε, but where distortion of every embedding of (X,d) is $\geq (1+\varepsilon)^{c' \cdot \log n / \log k}$
- **Theorem (upper bound)**: Metric spaces satisfying ε-4PC embed into trees with distortion $(1+\varepsilon)^{d \cdot \log n}$

Additional Results

- **Additive Distortion**:
  Metric spaces for which every k points embed into a tree with additive distortion δ embed into trees with additive distortion $\Theta(\delta \log n)$

  (main techniques needed already appear in [Gromov 87])

- **Distance Labeling**:
  ε-4PC metric space for which every $(1+\varepsilon)^{\log k}$-approximate distance labeling needs labels of size $n^{\Omega(1/k)}$

  (polylog labels → distortion $(1+\varepsilon)^{\Omega(\log \log n)}$)

  (similar to proof for additive distortion in [Gavoille,Ly 05])